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A POLYNOMIAL RING THAT IS JACOBSON RADICAL AND NOT NIL

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ABSTRACT

In [1] Amitsur conjectured that if a polynomial ring in one indeterminate is Jacobson radical then it is a nil ring. We shall construct an example disproving this conjecture.

0. Introduction

In [1] Amitsur proved that for every ring R the Jacobson radical $J(R[X])$ of the polynomial ring $R[X]$ in an indeterminate X over R is equal to $N[X]$ for a nil ideal N of R. He proved that if R is an algebra over an uncountable field, then $J(R[X])$ coincides with the upper nil radical of $R[X]$ (cf. also [2,8]) and conjectured (see also [3]) that this is true for all R. This conjecture is connected with Koethe's problem [6, 13] and some other problems of ring theory [4, 5, 10, 11, 12, 15]. We shall show that the conjecture does not hold. We construct an

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algebra over a countable field, the polynomial algebra in one indeterminate over which it is Jacobson radical but not a nil ring. In our proofs we shall use the main results of [14] (some of them are recalled in Section 1 and some others are extended in Section 3 to the form needed by us). We shall also apply Krempa's well known result [7] (cf. also [9,3]) which states that the ring $R[X]$ is Jacobson radical if and only if the rings $\mathcal{M}_n(R)$ of $n \times n$ matrices over R are nil for all n.

For a ring R, $R[X]$ will denote the ring of polynomials in an indeterminate X over R and $R[X, Y]$ the ring of polynomials in commuting indeterminates X, Y over R.

1. Throughout the paper K is a field and A is the algebra of polynomials in non-commuting indeterminates x, y, z over K. We denote by M the set of monomials in x, y, z and for each integer $n \geq 0$ M_n denotes the set of monomials of degree n. Thus $M_0 = \{1\}$ and for $n \geq 1$ the elements in M_n are of the form $x_1 \cdot \ldots \cdot x_n$, where $x_i \in \{x, y, z\}$. The K-subspace of A spaned by M_n will be denoted by H_n. Obviously $A = \bigoplus_{n=0}^{\infty} H_n$ and $H_n H_m = H_{n+m}$. Thus A is strongly graded in this way by the additive semigroup of non-negative integers.

Given $w \in M$ we denote by $d_x w$, $d_y w$ and $d_z w$ the x-degree, y-degree and z-degree of w, respectively. Obviously the degree of w, which we denote by dw, is equal to $d_x w + d_y w + d_z w$.

Given integers n_1, n_2, n_3 define

$$w(n_1, n_2, n_3) = \sum \{w \in M : d_x w = n_1, d_y w = n_2, d_z w = n_3\}$$

if all $n_i \geq 0$ and $w(n_1, n_2, n_3) = 0$ otherwise.

LEMMA 1.1 ([14], Lemma 7b): *For arbitrary integers p_1, p_2, p_3 and $n \leq p_1 + p_2 + p_3$, $w(p_1, p_2, p_3) = \sum \{w(n_1, n_2, n_3) w(p_1 - n_1, p_2 - n_2, p_3 - n_3) \mid n_1 + n_2 + n_3 = n\}$.*

Given a natural number n and a set $S \subseteq A$ let $B_n(S)$ denote the right ideal of A generated by the set $\bigcup_{k=0}^{\infty} M_{nk} S$, i.e., $B_n(S) = \sum_{k=0}^{\infty} M_{nk} S A$.

Let $m_1, m_2, \ldots$ be an increasing sequence of natural numbers such that each m_i divides m_{i+1} and let F_i be finite subsets of H_{m_i} and let $r_i = \operatorname{card} F_i$. In [14], Section 2, mappings R_i: $H_{m_i} \to H_{m_i}$ were introduced and studied. They have in particular the following properties.

LEMMA 1.2: (i) ([14], Theorem 4) *For every natural number l,*

$$R_{l+1}\left(H_{m_{l+1}} \cap \sum_{i=1}^{l} B_{m_{i+1}}(F_i)\right) = 0.$$

(ii) ([14], Theorem 11) *If* $m_1 > 10^8$ *and for each* $i > 0$, $m_{i+1} > m_i 2^{i+101}$ *and* $r_i = \operatorname{card} F_i < m_i^2 40^{-2}$, *then for every* $i > 0$ *there are integers* n_1, n_2, n_3 *such that* $n_1 + n_2 + n_3 = m_i$ *and* $R_i(w(n_1, n_2, n_3)) \neq 0$.

Now we shall prove

THEOREM 1.3: *Let* m_i *and* F_i *be as in Lemma 1.2(ii). For every* $k \geq 1$ *there are integers* n_1, n_2, n_3 *such that*

$$n_1 + n_2 + n_3 = m_k \quad \text{and} \quad w(n_1, n_2, n_3) \notin \sum_{i=1}^{\infty} B_{m_{i+1}}(F_i).$$

Proof: By Lemma 1.2(ii) for every $k \geq 1$ there are integers p_1, p_2, p_3 such that $p_1 + p_2 + p_3 = m_{k+1}$ and $R_{k+1}(w(p_1, p_2, p_3)) \neq 0$.

Suppose now that for all $n_1 + n_2 + n_3 = m_k$, $w(n_1, n_2, n_3) \in \sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$. Obviously $w(n_1, n_2, n_3) \in H_{m_k}$ and, for every $i \geq k+1$, $H_{m_k} \cap B_{m_{i+1}}(F_i) = 0$. Hence since all $B_{m_{i+1}}(F_i)$ are homogeneous right ideals of A, $w(n_1, n_2, n_3) \in \sum_{i=1}^{k} B_{m_{i+1}}(F_i)$. By Lemma 1.1,

$$w(p_1, p_2, p_3) = \sum \{ w(n_1, n_2, n_3) w(p_1 - n_1, p_2 - n_2, p_3 - n_3) \mid n_1 + n_2 + n_3 = m_k \},$$

so $w(p_1, p_2, p_3) \in \sum_{i=1}^{k} B_{m_{i+1}}(F_i)$. Hence by Lemma 1.2(i), $R_{k+1}(w(p_1, p_2, p_3)) = 0$, a contradiction. ∎

2. Let B be the algebra of polynomials in non-commuting indeterminates a, b, c over K and for each integer $n \geq 0$ let B_n be the K-subspace of B generated by all monomials w in a, b, c such that $d_a w + d_b w = n$, where for given $t \in \{a, b, c\}$ and a monomial w, $d_t w$ denotes the t-degree of w. Obviously $B = \bigoplus_{n=0}^{\infty} B_n$ and $B_n B_m = B_{n+m}$. Thus B is strongly graded in this way by the additive semigroup of non-negative integers. Note that the ideal $\langle ac, c^2 \rangle$ of B generated by ac and c^2 is homogeneous with respect to this grading. Thus $T = B/\langle ac, c^2 \rangle$ is a K-algebra strongly graded by the additive semigroup of non-negative integers with the homogeneous subspaces $T_n = B_n + \langle ac, c^2 \rangle / \langle ac, c^2 \rangle$.

Given non-negative integers n_1, n_2, n_3 denote by $\bar{w}(n_1, n_2, n_3)$ the image in T of the polynomial

$$\sum \{ m \mid m \text{ monomial in } a, b, c \text{ such that } d_a m = n_1, d_b m = n_2, d_c m = n_3 \}.$$

If at least one of n_1, n_2, n_3 is negative, we put $\bar{w}(n_1, n_2, n_3) = 0$.

In what follows, with a slight abuse of notation, we denote the images of a, b, c in T again by a, b, c. Then T is a K-algebra generated by $1, a, b, c$ satisfying relations $ac = 0$, $c^2 = 0$.

Note that there is an embedding of the algebra A into T such that $\phi(x) = a$, $\phi(y) = b$ and $\phi(z) = bc$. This allows us to assume that $A \subseteq T$ and $x = a$, $y = b$ and $z = bc$.

One easily checks the following

LEMMA 2.1: *The subalgebra $\bar{A}$ of T generated by x, y, z (i.e., the subalgebra of A consisting of polynomials in x, y, z with zero constant terms) is a right ideal of T and $T = A + cA$. Moreover, for each n, $H_n c \subseteq H_n$.*

Observe that if $w = w_1 \cdot \ldots \cdot w_m \neq 0$, where $w_i \in \{a, b, c\}$ and for some $2 \le i \le m$, $w_i = c$, then $w_{i-1} = b$. Applying this observation one easily gets

LEMMA 2.2: (a) *For every non-negative integer n, $T_n = H_n + cH_n$.*

(b) *Given integers n_1, n_2, n_3,*

$$\bar{w}(n_1, n_2 + n_3, n_3) = w(n_1, n_2, n_3) + cw(n_1, n_2 + 1, n_3 - 1).$$

We also shall need

LEMMA 2.3: (a) *If $h, h', g, g' \in A$ and $h + ch' = g + cg'$, then $h = g$ and $h' = g'$.*

(b) *If P is a right ideal of T contained in A and homogeneous in A, then $P + cP$ is a homogeneous right ideal of T.*

Now we shall prove

THEOREM 2.4: *Let $P \subseteq A$ be a right ideal of T which is homogeneous in A. If $(aX^2 + bX + c)^n \in (P + cP)[X]$, then $(x + yX + zY)^n \in P[X, Y]$.*

Proof: The coefficients of $(x + yX + zY)^n$ are $w(n_1, n_2, n_3)$, where n_i are non-negative integers such that $n_1 + n_2 + n_3 = n$. We have to show that all of them are in P. Clearly $m = n_1 + n_2 + 2n_3 \ge n$. Hence, since $P + cP$ is a right ideal of T, $(aX^2 + bX + c)^m \in (P + cP)[X]$. Given $0 \le s \le 2m$, the coefficient c_s at X^s in the polynomial $(aX^2 + bX + c)^m$ is equal to

$$\sum_{2m_1 + m_2 = s, m_1 + m_2 + m_3 = m} \bar{w}(m_1, m_2, m_3)$$

and it belongs to $P + cP$. Clearly $\bar{w}(m_1, m_2, m_3) \in T_{m_1 + m_2}$. Given $0 \le t \le m$, $t = m_1 + m_2$ if and only if $m_3 = m - t$, $m_1 = s - t$ and $m_2 = s - 2(s - t) = 2t - s$. Consequently the t-component of c_s with respect to the gradation of T is equal to $\bar{w}(s - t, 2t - s, m - t)$ and, by Lemma 2.3(b), it belongs to $P + cP$. For $s = 2n_1 + n_2 + n_3$ and $t = n = n_1 + n_2 + n_3$, we have $\bar{w}(s - t, 2t - s, m - t) = \bar{w}(n_1, n_2 + n_3, n_3)$. By Lemma 2.2(b), $\bar{w}(n_1, n_2 + n_3, n_3) = w(n_1, n_2, n_3) + cw(n_1, n_2 + 1, n_3 - 1)$.

Now applying Lemma 2.3(a) one gets that $w(n_1, n_2, n_3) \in P$. The result follows. ∎

3. We start this section with a lemma which slightly generalizes Lemma 1 of [14].

LEMMA 3.1: *Let G be a K-algebra strongly graded by the additive semigroup of non-negative integers with the homogeneous components G_i. For every $a = a_1 + a_2 + \cdots + a_t$, with $a_i \in G_i$, and each $n > t$ there is a set $G(a,n) \subseteq G_n$ such that $\operatorname{card} G(a,n) < nt \sup_{1\le i\le t} \dim_K G_i$ and $a^n = g_0 + g_1 a + \cdots + g_{n-2}a^{n-2}$ for some $g_0, \ldots, g_{n-2} \in G(a,n)(G_0 + \cdots + G_{t-1})$.*

Proof: Given $1 \le k \le n-1$ and $n-t \le l \le n-1$ let $w_{k,l} = \sum_{i_1+\cdots+i_k=l} a_{i_1}\cdots a_{i_k}$. The number of these $w_{k,l}$ is equal to $(n-1)t < nt$. Let

$$G(a,n) = \bigcup_{1\le k\le n-1}\ \bigcup_{n-t\le l\le n-1} w_{k,l}E_{n-l},$$

where E_{n-l} is a fixed K-basis of G_{n-l}. Clearly $G(a,n) \subseteq G_n$ and $\operatorname{card} G(a,n) < nt \sup_{1\le i\le t} \dim_K G_i$.

Now $a^n = \sum_{1\le i_1,\ldots,i_n\le t} a_{i_1}\cdots a_{i_n}$. Take any $1 \le i_1, \ldots, i_n \le t$. Since $i_1 < n$ and $i_1 + \cdots + i_n \ge n$, there exists $1 \le k \le n-1$ such that $i_1 + \cdots + i_k < n$ and $i_1 + \cdots + i_{k+1} \ge n$. Consequently

$$a^n = \sum_{k=1}^{n-1} \sum_{i_1+\cdots+i_k<n\le i_1+\cdots+i_{k+1}} a_{i_1}\cdots a_{i_{k+1}} \sum_{1\le i_{k+2},\ldots,i_n\le n} a_{i_{k+2}}\cdots a_{i_n}.$$

Note that for each $1 \le k \le n-1$ and fixed $i_1, \ldots, i_{k+1}$ with $i_1 + \cdots + i_k < n \le i_1 + \cdots + i_{k+1}$, we have $\sum_{1\le i_{k+2},\ldots,i_n\le t} a_{i_{k+2}}\cdots a_{i_n} = a^{n-k-1}$ (for $k = n-1$ the set of indices $1 \le i_{k+2}, \ldots, i_n \le t$ is empty and $n-k-1 = 0$; in this case we put both the sum and a^0 equal to 1). Thus

$$a^n = \sum_{k=1}^{n-1}\Big(\sum_{i_1+\cdots+i_k<n\le i_1+\cdots+i_{k+1}} a_{i_1}\cdots a_{i_{k+1}}\Big)a^{n-k-1}.$$

For fixed $1 \le k \le n-1$ we have

$$\sum_{i_1+\cdots+i_k<n\le i_1+\cdots+i_{k+1}} a_{i_1}\cdots a_{i_{k+1}} = \sum_{l=n-t}^{n-1}\Big(\sum_{i_1+\cdots+i_k=l} a_{i_1}\cdots a_{i_k}\Big)(a_{n-l} + \cdots + a_t)$$
$$= \sum_{l=n-t}^{n-1} w_{k,l}(a_{n-l} + \cdots + a_t).$$

Since G is a strongly graded K-algebra,

$$a_{n-l}+\cdots+a_t \in E_{n-l}(G_0+\cdots+G_{t-n+l}).$$

This implies that each $w_{k,l}(a_{n-l}+\cdots+a_t)$ is a linear combination over K of elements of $G(a,n)$ multiplied on the right by elements of

$$G_0+G_1+\cdots+G_{t-1}.$$

Hence indeed

$$a^n = g_0+g_1a+\cdots+g_{n-2}a^{n-2} \quad \text{for some } g_i \in G(a,n)(G_0+\cdots+G_{t-1}). \qquad \blacksquare$$

For a natural number j let $\mathcal{M}_j(T)$ denote the algebra of all $j\times j$ matrices over T. The algebra is strongly graded by the additive semigroup of non-negative integers with the homogeneous components $\mathcal{M}_j(T_n)$.

Given a subset $S\subseteq \mathcal{M}_j(T)$ we shall denote by $[S]$ the set of entries of elements in S.

LEMMA 3.2: *Suppose that $f=f_1+\cdots+f_t$, where $f_i\in\mathcal{M}_j(T_i)$. For every natural number $n>t$ there exists a set $S(f,n)\subseteq H_n$ with $\operatorname{card} S(f,n) < ntj^4 3^{t+3}$ satisfying $S(f,n)c\subseteq S(f,n)$ and such that for each $m>n$,*

$$[f^m]\subseteq (S(f,n)+cS(f,n))(T_0+\cdots+T_{t-1})[f^{m-n}]T.$$

Proof: Clearly $\dim_K \mathcal{M}_j(T_i) = j^2\dim_K T_i$. By Lemma 2.2(a), $T_n = H_n + cH_n$, so $\dim_K\mathcal{M}_j(T_i)\le j^2 2\dim_K H_i < j^2 3^{i+1}$. By Lemma 3.1 there is a set $G(f,n)\subseteq\mathcal{M}_j(T_n)$ with $\operatorname{card}G(f,n)<ntj^2 3^{t+1}$ and such that $f^n = g_0+g_1f+\cdots+g_{n-2}f^{n-2}$ for some $g_0,\dots,g_{n-2}\in G(f,n)\mathcal{M}_j(T_0+\cdots+T_{t-1})$, so $f^m = g_0f^{m-n}+g_1f^{m-n}f+\cdots+g_{n-2}f^{m-n}f^{n-2}$. Consequently f^m belongs to the right ideal of $\mathcal{M}_j(T)$ generated by $G(f,n)\mathcal{M}_j(T_0+\cdots+T_{t-1})f^{m-n}$. Since $T_n = H_n+cH_n$ and $G(f,n)\subseteq\mathcal{M}_j(T_n)$, there exists a subset $\bar S(f,n)$ of H_n such that $\operatorname{card}\bar S(f,n) < 2ntj^2 3^{t+1} j^2 < ntj^4 3^{t+2}$ and $G(f,n)\subseteq \mathcal{M}_j(\bar S(f,n)+c\bar S(f,n))$. Let $S(f,n)=\bar S(f,n)\cup \bar S(f,n)c$. Note that $G(f,n)\subseteq\mathcal{M}_j(S(f,n)+cS(f,n))$, so $[f^m]\subseteq(S(f,n)+cS(f,n))(T_0+\cdots+T_{t-1})[f^{m-n}]T$. Clearly $\operatorname{card}S(f,n)\le 2\operatorname{card}\bar S(f,n) < ntj^4 3^{t+3}$ and, by Lemma 2.1, $S(f,n)\subseteq H_n$. The result follows. $\blacksquare$

THEOREM 3.3: *Suppose that $f\in\mathcal{M}_j(T_1+\cdots+T_t)$ and let r,w be natural numbers such that $2t<r<w$. Then there is a set $S\subseteq H_r$ with $\operatorname{card} S < rj^4 3^{3t+3}$*

such that $B_w(S)$ is a right ideal of T and the ideal of T generated by $[f^{10w}]$ is contained in $B_w(S) + cB_w(S)$.

Proof: Let $S = \bigcup_{0\leq i\leq t} M_i S(f, r-i)$, where $S(f, r-i)$ are defined as in Lemma 3.2. Clearly $Sc \subseteq S$, which easily implies that $B_w(S)$ is a right ideal of T. We have $\operatorname{card} S < \sum_{i=0}^{t} 3^i (r-i) t j^4 3^{t+3} < r 3^{2t+3} j^4 t(t+1)$. Now $3^{2t+3} t(t+1) < 3^{3t+3}$, so $\operatorname{card} S < r j^4 3^{3t+3}$.

Note that for every natural number k, $T_{kw}(B_w(S) + cB_w(S)) \subseteq B_w(S) + cB_w(S)$. Hence, since $B_w(S)$ is a right ideal of T, to get that the ideal of T generated by $[f^{10w}]$ is contained in $B_w(S) + cB_w(S)$ it suffices to prove that $T_d[f^{10w}] \subseteq B_w(S) + cB_w(S)$ for $d < w$. Clearly $t < 2w - d < 10w$, so applying Lemma 3.2 for $n = 2w - d$ and $m = 10w$ we get that

$$T_d[f^{10w}] \subseteq T_d(S(f, 2w-d) + cS(f, 2w-d))(T_0 + \cdots + T_{t-1})[f^{8w+d}]T.$$

However, $T_d(S(f, 2w-d) + cS(f, 2w-d)) \subseteq T_{2w}$, so it suffices to show that for each $l < t$, $T_l[f^{8w+d}] \subseteq B_w(S) + cB_w(S)$. Clearly $t < r - l < 8w + d$, so applying Lemma 3.2 for $n = r - l$ and $m = 8w + d$ we get that

$$T_l[f^{8w+d}] \subseteq T_l(S(f, r-l) + cS(f, r-l))(T_0 + \cdots + T_{t-1})[f^{8w+d-r+l}]T.$$

From the definition of S and the fact that $H_l c \subseteq H_l$ it follows that $T_l(S(f, r-l) + cS(f, r-l)) \subseteq KS + cKS \subseteq B_w(S) + cB_w(S)$. Since $B_w(S)$ is a right ideal of T we get that $T_l[f^{8w+d}] \subseteq B_w(S) + cB_w(S)$. The result follows. ∎

In what follows $\bar{T}$ will denote the non-unital subalgebra of T generated by a, b, c and for every integer $n \geq 0$ we put $E_n = \bar{T} \cap (T_0 + \cdots + T_n)$.

Note that if $f \in \mathcal{M}_j(E_n)$, then $f^2 \in \mathcal{M}_j(T_1 + \cdots + T_{2n})$. Applying Theorem 3.3 and this observation one gets

COROLLARY 3.4: *Suppose that for $i = 1, 2, \ldots$, $f_i \in \mathcal{M}_{j_i}(E_{t_i})$ and m_i is an increasing sequence of natural numbers such that $4t_i < m_i$. Then there are $F_i \subseteq H_{m_i}$ with $\operatorname{card} F_i < m_i j_i^4 3^{6t_i+3}$ such that the ideal of T generated by $\bigcup_{i=1}^{\infty} [f_i^{20m_{i+1}}]$ is contained in $\sum_{i=1}^{\infty} B_{m_{i+1}}(F_i) + c\sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$ and $\sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$ is a right ideal of T.*

4. Now we are ready to prove the main result of the paper.

THEOREM 4.1: *If the field K is countable, then $\bar{T}$ contains an ideal I such that the polynomial algebra $(\bar{T}/I)[X]$ is Jacobson radical but not nil.*

Proof: Let $\mathcal{M}(\bar{T})$ be the algebra of countable matrices over $\bar{T}$ with a finite number of non-zero entries. Since K is countable, also $\mathcal{M}(\bar{T})$ is countable, i.e., $\mathcal{M}(\bar{T}) = \{f_1, f_2, \ldots\}$. Each f_i can be regarded as an element of $\mathcal{M}_{j_i}(E_{t_i})$ for some natural numbers t_i and j_i. There are natural numbers $m_1, m_2, \ldots$ satisfying

(i) $m_1 > 10^8$ and for each i, $m_{i+1} > m_i 2^{i+101}$,

(ii) each m_i divides m_{i+1},

(iii) $m_i > 40^2 3^{6t_i+3} j_i^4$.

Let I be the ideal of T generated by $\bigcup_{i=1}^{\infty}[f_i^{20m_{i+1}}]$. By Krempa's result ([7]) quoted in the introduction we get that $(\bar{T}/I)[X]$ is Jacobson radical. We shall show that it is not nil. Clearly $4t_i < m_i$, so applying Corollary 3.4 we get that there are $F_i \subseteq H_{m_i}$ with $\operatorname{card} F_i < m_i 3^{6t_i+3} j_i^4$ such that $I \subseteq P + cP$, where $P = \sum_{i=1}^{\infty} B_{m_{i+1}}(F_i)$ and P is a right ideal of T. By (iii), $3^{6t_i+3} j_i^4 < m_i 40^{-2}$, so $\operatorname{card} F_i < m_i^2 40^{-2}$. Hence from Theorem 1.3 we get that for every natural number k there are integers n_1, n_2, n_3 such that $n_1 + n_2 + n_3 = m_k$ and $w(n_1, n_2, n_3) \notin \sum_{i=1}^{\infty} B_{m_{i+1}}(F_i) = P$. This implies that for each k, $(x + yX + zY)^{m_k} \notin P[X, Y]$. Hence by Theorem 2.4, $(aX^2 + bX + c)^{m_k} \notin (P + cP)[X]$. However, $k \leq m_k$ and $I \subseteq P + cP$. Consequently, for every natural number k, $(aX^2 + bX + c)^k \notin I[X]$, so $(\bar{T}/I)[X]$ is not nil. ∎

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